1 Local solvability of generic over-determined PDE systems

Let $u = (u^1, \cdots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \cdots, x^p)$. Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u^{(m)}) = 0, \quad \lambda = 1, \cdots, \ell, \tag{1.1}$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. We assume (1.1) is over-determined, that is, $\ell > q$. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \cdots, j_r)$ with $1 \leq j_s \leq p$. The order of a multi-index J is denoted by $|J|$. By u^α_J we denote the $|J|$-th order partial derivative of u^α with respect to $x^{j_1}, \cdots, x^{j_{|J|}}$. For a smooth function $\Delta(x, u^{(m)})$, the total derivative of Δ with respect to x^i is the function in the arguments $(x, u^{(m+1)})$ defined by the chain rule:

$$D_i \Delta = \frac{\partial \Delta}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J| \leq m} \frac{\partial \Delta}{\partial u^\alpha_J} u^\alpha_{Ji},$$

where Ji dentes the multi-index $(j_1, \cdots, j_{|J|}, i)$. Compatibility conditions are those equations obtained from (1.1) by differentiation and algebraic operations, that is, the ideal generated by Δ and the total derivatives of Δ. By (1.1)'s being generic we shall mean that the compatibility conditions determine all the partial derivatives of u of a sufficiently high order, say $k (k \geq m)$, as functions of derivatives of lower order, namely,

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}), \tag{1.2}$$

for all multi-index K with $|K| = k$, and for all $\alpha = 1, \cdots, q$. This is the case where there exists a system of compatibility conditions that satisfies the non-degeneracy hypothesis of the implicit function theorem so that the system is solvable for all u_K's with $|K| = k$ in terms of lower order derivatives. If this is the case (1.2) is called a complete system of order k and (1.1) is said to admit prolongation to a complete system (1.2). Now we consider the ring of smooth functions in the arguments $(x, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \cdots)$. For each non-negative integer r let $\Delta^{(r)}$ be the algebraic ideal generated by Δ and the total derivatives of Δ up to order r, where $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ as in (1.1). Suppose that the complete system (1.2) is obtained from $\Delta^{(r)}$. Let $J^{k-1}(X, U)$ be the space of $(k-1)$th jets $(x, u^{(k-1)})$. Let $\mathcal{S} \subset J^{k-1}(X,U)$ be the common zero set of those functions in the arguments $(x, u^{(k-1)})$ that are elements of $\Delta^{(r)}$. We assume $\mathcal{S}$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge x^p \neq 0$. Then there exist disjoint sets of indices

$$A := \{(a, I)\} \quad \text{and } B := \{(b, J)\},$$

where $a, b \in \{1, \cdots, q\}$, I and J are multi-indices of order $\leq k-1$ so that $\mathcal{S}$ is the graph

$$u^b_J = \Phi^b_J(x, u^a_I :), \quad \text{for all } (b, J) \in B. \tag{1.3}$$

We take $(x, u^a_I : (a, I) \in A)$, as local coordinates of $\mathcal{S}$. Observe that if $u = u(x)$ is a solution of (1.1) then its $(k-1)$th jet graph $(x, u^{(k-1)}(x))$ is contained in $\mathcal{S}$ and for each $(a, I) \in A$, we have

$$du^a_I(x) = \begin{cases} \sum_{i=1}^p u^a_{Ii}(x) dx^i & \text{for } \ |I| \leq k-2, \\ \sum_{i=1}^p H^a_{Ii}(x, u^{(k-1)}) dx^i & \text{for } \ |I| = k-1, \end{cases}$$

where H's are as in (1.2).
Substituting (1.3) for all u^b_J with $(b, J) \in B$ we obtain

$$du^a_I(x) = \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L(x)) dx^i,$$

where all the indices (α, L) are in A. Thus on $\mathcal{S}$ we define independent 1-forms

$$\theta^a_I := du^a_{Ii} - \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L : (\alpha, L) \in A) dx^i, \tag{1.4}$$

for all $(a, I) \in A$. Then the smooth solutions of (1.1) are in one-to-one correspondence with the smooth integral manifolds of the Pfaffian system (1.4). Let $\theta = \{\theta^a_I : (a, I) \in A\}$. We set

$$d\theta^a_I \equiv \sum_{i<j} T^a_{Iij}(x, u^\alpha_J : (\alpha, J) \in A) dx^i \wedge dx^j, \qquad \mod \theta.$$

If all of $T := \{T^a_{Iij} : (a, I) \in A\}$ are identically zero then $\mathcal{S}$ is foliated by integral manifolds by the Frobenius theorem. Otherwise, we pull back the Pfaffian system (1.4) to a subset $\mathcal{S}' \subset \mathcal{S}$, where $\mathcal{S}'$ is the common zero set of $\{\Delta, T\}$ and their compatibility conditions that are functions in $(x, u^{(k-1)})$. We assume that $\mathcal{S}'$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge dx^p \neq 0$. Making use of the following theorem we further check the integrability of (1.4) on $\mathcal{S}'$.

Theorem 1.1 *Let M be a smooth manifold of dimension n. Let $\theta := (\theta^1, \cdots, \theta^s)$ be a set of independent 1-forms on M and $\mathcal{D} :=< \theta >^{\perp}$ be the $(n-s)$ dimensional plane field annihilated by θ. Suppose that N is a submanifold of M of dimension $n-r$ defined by $T_1 = \cdots = T_r = 0$, where T_i are smooth real-valued functions of M such that $dT_1 \wedge \cdots \wedge dT_r \neq 0$ on N. Then the following are equivalent :*

(i) *$\mathcal{D}$ is tangent to N.*

(ii) $dT_j \equiv 0 \mod \theta^1, \ldots, \theta^s$ *on N, $1 \leq j \leq r$,*

(iii) *$i^*\theta^1, \cdots, i^*\theta^s$ have rank $s-r$, where $i : N \hookrightarrow M$ is the inclusion.*

Proof. (i) $\Leftrightarrow$ (ii)

Let $P \in N$. Then

$\mathcal{D}_P$ is tangent to N.
$\Leftrightarrow < \theta^1, \cdots, \theta^s >^{\perp} \subset < dT_1, \cdots, dT_r >^{\perp}$ at P.
$\Leftrightarrow dT_i \in < \theta^1, \cdots, \theta^s >$ at P for each $i = 1, \cdots, r$.
$\Leftrightarrow dT_i \equiv 0, \mod \theta$.

(ii) $\Leftrightarrow$ (iii)

Assuming (ii), we have $dT_j = \sum_{k=1}^{s} a_{jk}\theta^k$ on N, $j = 1, \ldots, r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, the matrix (a_{jk}) has rank r. Then we have

$$0 = i^*dT_j = \sum_{k=1}^{s} a_{jk} i^*\theta^k, \ 1 \leq j \leq r.$$

This means that $i^*\theta^1, \cdots, i^*\theta^s$ have rank no greater than $k-r$. Since N is of dimension $n-r$, $i^*\theta^1, \cdots, i^*\theta^s$ have rank no less than $k-r$. Therefore we obtain (iii).
Conversely, (iii) implies that (after suitable index changes) $i^*\theta^{s-r+l} = \sum_{j=1}^{s-r} b_{lj} i^*\theta^j$, $1 \leq l \leq r$ for some functions on N. Then $i^*(\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j) = 0$, $1 \leq l \leq r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, we have $\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j = \sum_j c_{lj} dT_j$, $1 \leq l \leq r$ on N, where c_{ij} are functions on N and the matrix (c_{ij}) is nonsingular. Therefore we can solve dT_j in terms of θ^k's. □

Thus $\mathcal{S}'$ is foliated by integral manifolds if $dT^a_{Iij} \equiv 0 \mod \theta$ on $\mathcal{S}'$. Otherwise, we repeat the same argument, eventually to reach a submanifold $\tilde{\mathcal{S}}$ of dimension $\leq p$. If $\tilde{\mathcal{S}}$ is of dimension p then the Frobenius integrability is simply

$$i^*\theta = 0,$$

where $i : \tilde{\mathcal{S}} \to \mathcal{S}$ is the inclusion map. If the dimension of $\tilde{\mathcal{S}}$ is less than p, no solution exists.

As for the prolongation to a complete system we refer to [**Han**].
Our standard reference on exterior differential system are [**Br**] and [**BCGGG**].

References

[**Br**] R. Bryant, *Nine lectures on exterior differential systems*, MSRI Lecture Note, 2002.

[**BCGGG**] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems*, Springer-Verlag, Berlin, 1991.

[**Han**] C. K. Han, *Equivalence problem and complete system of finite order*, J. Korean Math. Soc. 37(2000), 225-243.